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POMs analysis of randomly vibrating systems obtained from Karhunen–Loève expansion

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Abstract

The Karhunen–Loève (KL) decomposition establishes that a 2D random field can be expanded as a series involving a sequence of deterministic orthogonal functions with orthogonal random coefficients. The proper orthogonal decomposition (POD) method consists in detecting spatially coherent modes in the dynamics of a spatio-temporally varying system by diagonalizing the spatial correlation function given by an averaging operator. The KL expansion is applied here to the responses of randomly excited vibrating systems with a view to performing a POD in separated-variables (time and space) form.

Discrete and continuous mechanical systems are considered in this study as well as stationary and transient (non-stationary) responses. An averaging operator involving time and ensemble averages is proposed to draw up the POD in separated-variables form from the associated KL expansion. The result obtained using this approach agrees with the classical POD in the case of deterministic or ergodic random signals. The associated proper orthogonal modes are interpreted in case of linear and nonlinear vibrating systems subjected to white noise excitation in terms of normal modes.

1. Introduction

In several domains of engineering science, it can be useful to have a second-moment characterization of a random field in terms of uncorrelated random variables. An expansion of this kind can be found in the literature where it is known as the Karhunen–Loève (KL) expansion. The basis functions in this expansion, which are also called Karhunen–Loève modes (KLMs), are the eigenfunction solutions of the Fredholm integral equation, the kernel of which is the autocorrelation (or autocovariance) function of the random field under study. The main properties of the KL expansion are the orthogonality of the eigenfunctions and the random variables taken as coefficients and the error-minimizing property. This expansion was developed in the 1940s by several authors (Kosambi, 1943; Loève, 1945; Karhunen, 1947; Kac and Siegert,

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1947; Obukhov, 1954; Pougachev, 1953). It was subsequently investigated and used in many branches of engineering science. Depending on the properties of the random field under study, the use of the expansion, and/or the field of application, this expansion has been given under different names such as principal component analysis (PCA), proper orthogonal decomposition (POD), and singular value decomposition (SVD) [1].

In the field of random mechanics the KL expansion has been intensively used. It is one of the main tools used to develop the stochastic finite elements method [2]. It is also one of the techniques used to simulate random fields when they are specified by its covariance function and its marginal density probability [3,4]. Some methods of characterizing stochastic dynamic responses have also been developed in which the KL expansion is used to account for the excitation process [5–7]. In studies of this kind, the KL expansion is obtained starting with an analytical form of the covariance function, and neither simulated nor experimental data are required.

When the term POD is used to denote an expansion, it generally refers to a characterization of the signal based on experimental data. As defined in Ref. [1], the POD is *a multi-variate statistical method that aims at obtaining a compact representation of the data*. The POD also involves detecting spatially coherent modes in the dynamics of a spatio-temporally varying system by diagonalizing the spatial covariance function of data with respect to an averaging operation. In the case of random fields, the averaging operation is taken to be the ensemble average and the POD expansion is called the KL expansion. In the case of spatio-temporal data (not necessarily random ones), the averaging operation is focuses typically on the time average. As illustrated in Refs. [8,9], this is not the only possibility and, when data correspond to a random-response process, the stationarity in time and the ergodicity are required to relate the time average to the ensemble average or mean operator. The POD has been generally used, combined with the Galerkin method, for model reduction purposes. In vibration analysis the KLMs, or proper orthogonal modes (POMs), advantageously replaces the linear normal modes (LNMs) of the underlying linear system (see for example Refs. [10–13]).

The physical interpretation of POMs has also been investigated. These modes have been related to the LNMs of multi-modal free responses of discrete symmetrical systems when the mass matrix has the form mI and for lightly damped systems [14,15]. In these cases, time averaging has been used as the averaging operation in the POD method. In Refs. [15,20], it was also established that the POMs are the principal axes of the potential bounding ellipsoid. This result has been extended to distributed parameter systems [16,17] and used to extract mode shapes by applying the POD on the measured response data [18]. Conservative linear systems (discrete and continuous) under random excitation have been studied in Ref. [19]. It was established that POMs converge to the LNMs if the mass distribution is known and if each mode is excited with a random process with a convergent Fourier transform. Here again the time average was used as the averaging operation in the POD method. One of the consequences of this choice is that the eigenvalues associated with the POMs can depend on the temporal trajectories. Linear discrete mechanical systems subjected to Gaussian white-noise excitation have been addressed in Ref. [15]. In this case ensemble averaging has been used as the averaging operation in the POD method (KL expansion method). It was shown in the latter study that the KLM of the stationary responses can be obtained by solving the algebraic Lyapunov equation and that these responses are related to the principal axis of the ellipsoids defined by the contours of the joint probability density of the displacement response.

Here we show how the KL expansion can be used to obtain a POD of randomly excited vibrating systems in terms of separate variables (time and space) and separate characteristics (random and deterministic). Discrete and continuous mechanical systems are studied in this context as well as transient (non-stationary) and stationary responses. An averaging operator involving time and ensemble averaging is presented for obtaining the POD in separate form from the associated KL expansion. This approach corresponds to the classical POD used in the case of deterministic or ergodic random signals. As in the classical POM approach, there are two ways of constructing the expansion from data: the direct and the snapshot methods. Each method has its own field of application, as will be discussed below, although this paper uses only the direct method. The associated POMs are interpreted, in the case of both linear and nonlinear vibrating systems subjected to white-noise excitation, in terms of normal modes.

2. KL expansion

The mathematical formulation used here to present the KL expansion was based on those used in Refs. [21–23]. It involves concepts of a rather abstract and mathematical kind which are not conventionally used in the field of engineering science.

Let $\mathscr{D}$ be a compact subset of $\mathbb{R}^l$ and $\{\mathbf{X}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$ a stochastic vector field defined on a probability space $(\Omega,\mathscr{F},P)$ with values in $\mathbb{R}^d$. This random field is an l-parameter family on real-valued vector, $\mathbf{X}(\mathbf{z},\theta)$, for $(\mathbf{z},\theta)\in\mathscr{D}\times\Omega$ where $\mathscr{D}$ denotes the space of physical variables and Ω the space of random events.

Let $L^2(\Omega,\mathbb{R}^d)$ be the Hilbert space of the second-order random vector variables defined on the probability space $(\Omega,\mathscr{F},P)$ with the inner product

$$\langle\mathbf{Y},\mathbf{Z}\rangle_\Omega=\int_\Omega\langle\mathbf{Y}(\theta),\mathbf{Z}(\theta)\rangle\,\mathrm{d}P(\theta)=E(\langle\mathbf{Y},\mathbf{Z}\rangle),\tag{1}$$

where $\langle.,.\rangle$ denotes the Euclidian inner product in $\mathbb{R}^d$, $\mathrm{d}P(\theta)$ is the probability measure, and $E(.)$ denotes the mean, or ensemble average, with respect to the probability measure P.

Let $L^2(\mathscr{D},\mathbb{R}^d)$ be the Hilbert space of the square integrable vector functions defined on $\mathscr{D}$ with the inner product

$$\langle\mathbf{f},\mathbf{g}\rangle_\mathscr{D}=\int_\mathscr{D}\langle\mathbf{f}(\mathbf{z}),\mathbf{g}(\mathbf{z})\rangle\,\mathrm{d}\mathbf{z}.\tag{2}$$

We will assume that the random field $\{\mathbf{X}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$ satisfies the following assumptions:

I: $\{\mathbf{X}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$ is a second-order random field, i.e., $\forall\mathbf{z}\in\mathscr{D}$,

$$E(\langle\mathbf{X}(\mathbf{z}),\mathbf{X}(\mathbf{z})\rangle)<\infty,\ \text{or}\ (\mathbf{X}(\mathbf{z})\in L^2(\Omega,\mathbb{R}^d)),$$

II: $\{\mathbf{X}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$ is continuous in quadratic mean.

The random field can be regarded as a curve in either $L^2(\Omega,\mathbb{R}^d)$ or $L^2(\mathscr{D},\mathbb{R}^d)$.

Under the previous assumptions, it can be shown that the covariance (or autocovariance) matrix function of the random field $\{\mathbf{X}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$,

$$\mathbf{C}_\mathbf{X}(\mathbf{z}_1,\mathbf{z}_2)=E((\mathbf{X}(\mathbf{z}_1)-\mathbf{m}_\mathbf{X}(\mathbf{z}_1))(\mathbf{X}(\mathbf{z}_2)-\mathbf{m}_\mathbf{X}(\mathbf{z}_2))^{\mathrm{T}}),\tag{3}$$

(where $\mathbf{m}_\mathbf{X}(\mathbf{z})=E(\mathbf{X}(\mathbf{z}))$ denotes the vector mean-function of the random field) defines a continuous self-adjoint Hilbert–Schmidt operator, Q, on $L^2(\mathscr{D},\mathbb{R}^d)$ by

$$(Q\boldsymbol{\psi})(\mathbf{z})=\int_\mathscr{D}\mathbf{C}_\mathbf{X}(\mathbf{z},\mathbf{z}')\boldsymbol{\psi}(\mathbf{z}')\,\mathrm{d}\mathbf{z}'\quad\text{for}\ \boldsymbol{\psi}\in L^2(\mathscr{D},\mathbb{R}^d).$$

The associated eigenvalue problem, $Q\boldsymbol{\psi}=\lambda\boldsymbol{\psi}$, has a countable number of eigenvalues $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k\geqslant\cdots$, and the associated normalized eigenfunctions constitute an orthonormal basis, $\{\boldsymbol{\psi}_k\}_{k\geqslant1}$, of $L^2(\mathscr{D},\mathbb{R}^d)$ (i.e., $\langle\boldsymbol{\psi}_{k_1},\boldsymbol{\psi}_{k_2}\rangle_\mathscr{D}=\delta_{k_1,k_2}$).

The eigenfunctions $\{\boldsymbol{\psi}_k\}_{k\geqslant1}$ can be used as a basis for decomposing the random field $\{\mathbf{X}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$ such as

$$\mathbf{X}(\mathbf{z},\theta)-\mathbf{m}_\mathbf{X}(\mathbf{z})=\sum_{k=1}^{\infty}\xi_k(\theta)\boldsymbol{\psi}_k(\mathbf{z})\tag{4}$$

(the equality is in $L^2(\Omega,\mathbb{R}^d)$) where $\xi_1,\xi_2,\ldots,\xi_k,\ldots$ are scalar zero-mean uncorrelated random variables, i.e.,

$$E(\xi_k)=0\quad\text{and}\quad E(\xi_{k_1}\xi_{k_2})=\lambda_{k_1}\delta_{k_1,k_2}\tag{5}$$

given by

$$\xi_k(\theta)=\int_\mathscr{D}\langle\mathbf{X}(\mathbf{z},\theta)-\mathbf{m}_\mathbf{X}(\mathbf{z}),\boldsymbol{\psi}_k(\mathbf{z})\rangle\,\mathrm{d}\mathbf{z}(=\langle\mathbf{X}(\theta)-\mathbf{m}_\mathbf{X},\boldsymbol{\psi}_k\rangle_\mathscr{D}).\tag{6}$$

The eigenvalues, λ_k, are related to the "energy" of the random field by means of the following relation:

$$E(\|\mathbf{X}-\mathbf{m}_{\mathbf{X}}\|^2_{\mathscr{D}}) = \sum_{k=1}^{\infty} \lambda_k. \tag{7}$$

For every positive integer p, and for any arbitrary orthogonal basis, $(\tilde{\boldsymbol{\psi}}_k)$, of $L^2(\mathscr{D},\mathbb{R}^d)$ where $\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_k, \ldots$ are scalar random variables given by

$$\tilde{\xi}_k(\theta) = \int_{\mathscr{D}} \langle \mathbf{X}(\mathbf{z},\theta) - \mathbf{m}_{\mathbf{X}}(\mathbf{z}), \tilde{\boldsymbol{\psi}}_k(z)\rangle \, \mathrm{d}z,$$

the following relation holds

$$E\left(\left\|\mathbf{X}-\mathbf{m}_{\mathbf{X}}-\sum_{k=1}^{p}\xi_k\boldsymbol{\psi}_k\right\|^2_{\mathscr{D}}\right) = \sum_{k=1}^{p}\lambda_k \leqslant E\left(\left\|\mathbf{X}-\mathbf{m}_{\mathbf{X}}-\sum_{k=1}^{p}\tilde{\xi}_k\tilde{\boldsymbol{\psi}}_k\right\|^2_{\mathscr{D}}\right). \tag{8}$$

Expansion (4) is the KL expansion, or decomposition, of the random field $\{\mathbf{X}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$. The set $\{\boldsymbol{\psi}_k\}_{k\geqslant 1}$ of deterministic functions will be referred to here as the KL basis. The kth eigenvalue gives the average energy in the direction of kth KL basis vector and the KL basis is the optimum basis for expressing the random field, i.e., no other basis contains more energy in fewer elements.

3. POD using the KL expansion

We now examine how to use the KL expansion to extract models which capture the behaviour of a random vibrating system. Before starting, it is important to note that in random vibrations, the physical domain $\mathscr{D}$, in which the random field based on the displacement field $\{\mathbf{u}(\mathbf{z})\}_{\mathbf{z}\in\mathscr{D}}$ is defined, depends on the characteristics of the structure. In the case of discrete mechanical systems, $\mathscr{D}=\mathscr{D}_T\subset\mathbb{R}$ ($l=1$) and $\mathscr{D}_T$ usually defines the time interval of interest and, without any loss of generality, we will assume in what follows that $\mathscr{D}_T=[0,T]$ with $T\in\mathbb{R}^+$. In the case of continuous mechanical systems the space domain is $\mathscr{D}=\mathscr{D}_T\times\mathscr{D}_{\mathbf{x}}$, where $\mathscr{D}_{\mathbf{x}}\subset\mathbb{R}^p$(with $p=1,2$, or 3, i.e., $l=2,3$, or 4).

In dynamics problems we usually want to develop the displacement field into a series in the separated-variables form

$$\mathbf{u}(t,\mathbf{x},\theta) = \sum_{k=1}^{\infty} a_k(t,\theta)\boldsymbol{\phi}_k(\mathbf{x}), \tag{9}$$

where $\boldsymbol{\phi}_k$ are deterministic $\mathbb{R}^d$-valued functions and $\{a_k(t)\}_{t\in\mathscr{D}_T}$ are scalar random processes. If the functions $\boldsymbol{\phi}_k$ and/or the random processes $\{a_k(t)\}_{t\in\mathscr{D}_T}$ satisfy some orthogonal and optimality properties, expansion (9) will be called the POD and $\boldsymbol{\phi}_k$ and λ_k will be called POMs and proper orthogonal values (POVs), respectively. The orthogonality properties can serve, for example, to construct reduced-order models and to extract "modal" properties.

It is important to realize that Eq. (9) is not the standard KL expansion given in Eq. (4). In Eq. (4), $\mathbf{z}=(t,\mathbf{x})$ and in Eq. (9), t and $\mathbf{x}$ are treated differently. In Eq. (9), we take into account the fact that we are dealing with a dynamic system. To obtain a POD in the form of Eq. (9), where the variables t and $\mathbf{x}$ play different roles, we have to treat them differently.

3.1. Time-stationary case

If $\{\mathbf{u}(t,\mathbf{x})\}_{(t,\mathbf{x})\in\mathscr{D}_T\times\mathscr{D}_{\mathbf{x}}}$ is time-stationary, its covariance matrix function satisfies the property $\mathbf{C}_{\mathbf{u}}(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) = \mathbf{C}_{\mathbf{u}}(t_1-t_2,\mathbf{x}_1,\mathbf{x}_2)$ and the (spatial) covariance matrix function $\mathbf{C}_{\mathbf{X}}(t,\mathbf{x}_1,t,\mathbf{x}_2)=\mathbf{C}_{\mathbf{u}}(0,\mathbf{x}_1,\mathbf{x}_2)$ does not depend on time t.

For fixed $t\in\mathscr{D}_T$, the KL theory described in Section 2 can be applied to the random field $\{\mathbf{u}(t,\mathbf{x})\}_{\mathbf{x}\in\mathscr{D}_{\mathbf{x}}}$, yielding the following expansion (in $L^2(\Omega,\mathbb{R}^d)$):

$$\mathbf{u}(t,\mathbf{x},\theta) - \mathbf{m}_{\mathbf{u}}(\mathbf{x}) = \sum_{k=1}^{\infty}\xi_k(t,\theta)\boldsymbol{\psi}_k(\mathbf{x}), \tag{10}$$

where $\boldsymbol{\psi}_k$ are the eigenfunctions of the Hilbert–Schmidt operator (on $L^2(\mathscr{D}_{\mathbf{x}}, \mathbb{R}^d)$)

$$((Q\boldsymbol{\psi}_k)(\mathbf{x}) =) \int_{\mathscr{D}_{\mathbf{x}}} \mathbf{C}_{\mathbf{u}}(0, \mathbf{x}, \mathbf{x}')\boldsymbol{\psi}_k(\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_k \boldsymbol{\psi}_k(\mathbf{x}) \tag{11}$$

and the zero-mean uncorrelated random coefficients $\xi_k(t)$ (i.e. $E(\xi_{k_1}(t)\xi_{k_2}(t)) = \lambda_{k_1}\delta_{k_1,k_2}$) are given by

$$\xi_k(t,\theta) = \int_{\mathscr{D}_{\mathbf{x}}} \langle \mathbf{u}(t,\mathbf{x},\theta) - \mathbf{m}_{\mathbf{u}}(\mathbf{x}), \boldsymbol{\psi}_k(\mathbf{x})\rangle\,\mathrm{d}\mathbf{x} (= \langle \mathbf{u}(t,\theta) - \mathbf{m}_{\mathbf{u}}, \boldsymbol{\psi}_k\rangle_{\mathscr{D}_{\mathbf{x}}}). \tag{12}$$

Due to the stationarity property, the mean function of the random field, the eigenfunctions, $\boldsymbol{\psi}_k$, and the eigenvalues, λ_k, do not depend on time. Expansion (10) defines a POD of the time-stationary random field $\{\mathbf{u}(t,\mathbf{x})\}_{(t,x)\in\mathscr{D}_T\times\mathscr{D}_{\mathbf{x}}}$ in the form of Eq. (9) and the optimality relation (8) reduces to

$$E\left(\left\|\mathbf{u}(t) - \mathbf{m}_{\mathbf{u}} - \sum_{k=1}^{p} \xi_k(t)\boldsymbol{\psi}_k\right\|^2_{\mathscr{D}_{\mathbf{x}}}\right) = \sum_{k=1}^{p} \lambda_k \leqslant E\left(\left\|\mathbf{u}(t) - \mathbf{m}_{\mathbf{u}} - \sum_{k=1}^{p} \tilde{\xi}_k(t)\tilde{\boldsymbol{\psi}}_k\right\|^2_{\mathscr{D}_{\mathbf{x}}}\right). \tag{13}$$

3.2. Non-stationary case

When $\{\mathbf{u}(t,\mathbf{x})\}_{(t,\mathbf{x})\in\mathscr{D}_T\times\mathscr{D}_{\mathbf{x}}}$ is not time stationary, the time variable has to be included (in the same way as the random parameter θ) in the averaging operation.

We consider the Hilbert space $L^2(\mathscr{D}_T \times \Omega, \mathbb{R}^d)$ with the inner product

$$\langle \mathbf{Y}, \mathbf{Z}\rangle_{\mathscr{D}_T\times\Omega} = \frac{1}{T}\int_0^{\mathrm{T}} E(\langle \mathbf{Y}(t), \mathbf{Z}(t)\rangle)\,\mathrm{d}t = \mathbb{E}(\langle \mathbf{Y}, \mathbf{Z}\rangle).$$

The random process $\{\mathbf{u}(t,\mathbf{x})\}_{(t,\mathbf{x})\in\mathscr{D}_T\times\mathscr{D}_{\mathbf{x}}}$ can now be regarded as a curve in either $L^2(\mathscr{D}_T \times \Omega, \mathbb{R}^d)$ or $L^2(\mathscr{D}_{\mathbf{x}}, \mathbb{R}^d)$ and it will be denoted in what follows by $\{\{\mathbf{u}(\mathbf{x})\}\}_{\mathbf{x}\in\mathscr{D}_{\mathbf{x}}}$ to point out that variables t and θ are both included in the averaging operation.

With this new averaging operation, the mean and (spatial) autocovariance matrix functions of the random field $\{\{\mathbf{u}(\mathbf{x})\}\}_{\mathbf{x}\in\mathscr{D}_{\mathbf{x}}}$ are defined by

$$\begin{cases} \mathbf{m}_{\mathbf{u}}(\mathbf{x}) = \mathbb{E}(\mathbf{u}(\mathbf{x})), \\ \mathbb{C}_{\mathbf{u}}(\mathbf{x},\mathbf{x}') = \mathbb{E}((\mathbf{u}(\mathbf{x}) - \mathbf{m}_{\mathbf{u}}(\mathbf{x}))(\mathbf{u}(\mathbf{x}') - \mathbf{m}_{\mathbf{u}}(\mathbf{x}'))^{\mathrm{T}}). \end{cases}$$

Provided assumptions I and II (Section 2) hold with the new averaging operation, the random field $\{\{\mathbf{u}(\mathbf{x})\}\}_{\mathbf{x}\in\mathscr{D}_{\mathbf{x}}}$ can be expanded into (the equality is in $L^2(\mathscr{D}_T \times \Omega, \mathbb{R}^d)$)

$$\mathbf{u}(t,\mathbf{x},\theta) - \mathbf{m}_{\mathbf{u}}(\mathbf{x}) = \sum_{k=1}^{\infty} \xi_k(t,\theta)\boldsymbol{\psi}_k(\mathbf{x}), \tag{14}$$

where the $\boldsymbol{\psi}_k$ solve the eigenvalue problem

$$\int_{\mathscr{D}_{\mathbf{x}}} \mathbb{C}_u(\mathbf{x},\mathbf{x}')\boldsymbol{\psi}_k(\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_k\boldsymbol{\psi}_k(\mathbf{x}) \tag{15}$$

and $\xi_1, \xi_2, \ldots, \xi_m, \ldots$ are scalar zero-mean random processes given by

$$\xi_k(t,\theta) = \int_{\mathscr{D}_{\mathbf{x}}} \langle \mathbf{u}(t,\mathbf{x},\theta) - \mathbf{m}_{\mathbf{u}}(\mathbf{x}), \boldsymbol{\psi}_k(\mathbf{x})\rangle\,\mathrm{d}\mathbf{x} \tag{16}$$

which satisfy the orthogonality properties $\mathbb{E}(\xi_{k_1}\xi_{k_2}) = \lambda_{k_1}\delta_{k_1,k_2}$.

Expansion (14) defines the POD of a (non-stationary) random field $\{\{\mathbf{u}(\mathbf{x})\}\}_{\mathbf{x}\in\mathscr{D}_{\mathbf{x}}}$ (or, equivalently, of $\{\mathbf{u}(t,\mathbf{x})\}_{(t,x)\in\mathscr{D}_T\times\mathscr{D}_{\mathbf{x}}}$) and the optimality relation (8) reads as follows

$$\mathbb{E}\left(\left\|\mathbf{u} - \mathbf{m}_{\mathbf{u}} - \sum_{k=1}^{p} \xi_k\boldsymbol{\psi}_k\right\|^2_{\mathscr{D}_{\mathbf{x}}}\right) = \sum_{k=1}^{p} \lambda_k \leqslant \mathbb{E}\left(\left\|\mathbf{u} - \mathbf{m}_{\mathbf{u}} - \sum_{k=1}^{p} \tilde{\xi}_k\tilde{\boldsymbol{\psi}}_k\right\|^2_{\mathscr{D}_{\mathbf{x}}}\right). \tag{17}$$

3.3. Some comments

1. When working with the POD of a random field using the KL theory it is necessary to define clearly the averaging procedure.
2. The existence of the POD based on KL expansion, as described in Eqs. (14)–(16), does not require any assumptions about stationarity or ergodicity properties.
3. The POD based on KL expansion described in Eqs. (14)–(16) usually depends on the time parameter T. The role of T has to be carefully examined when the random field is not stationary.
4. If the random field $\{\mathbf{u}(t,\mathbf{x})\}_{(t,\mathbf{x})\in\mathscr{D}_T\times\mathscr{D}_\mathbf{x}}$ is time stationary, expansion (14) coincides with expansion (10). Moreover, the expansion does not depend on the time parameter T and the energy relations (13) and (17) are equivalent.

3.4. Discrete mechanical systems

In the discrete mechanical case, i.e., when the physical space is of the form $\mathscr{D}=\mathscr{D}_T$, expansions (10) and (14) reduce to

$$\mathbf{u}(t,\theta)-\mathbf{m}_\mathbf{u}^d=\sum_{k=1}^{n}\xi_k(t,\theta)\boldsymbol{\psi}_k, \tag{18}$$

where the $\boldsymbol{\psi}_k$ are now constant vectors which solve the eigenvalue problem

$$\mathbf{C}_\mathbf{u}^d\boldsymbol{\psi}_k=\lambda_k\boldsymbol{\psi}_k \tag{19}$$

with

$$\begin{cases}\mathbf{m}_\mathbf{u}^d=E(\mathbf{u}(t))\\ \mathbf{C}_\mathbf{u}^d=E((\mathbf{u}(t)-\mathbf{m}_\mathbf{u}^d)(\mathbf{u}(t)-\mathbf{m}_\mathbf{u}^d)^{\mathrm{T}})\end{cases}\quad\text{if } \{\mathbf{u}(t)\}_{t\in\mathscr{D}_T} \text{ is stationary} \tag{20}$$

and

$$\begin{cases}\mathbf{m}_\mathbf{u}^d=\mathbb{E}(\mathbf{u}(t))\\ \mathbf{C}_\mathbf{u}^d=\mathbb{E}(\mathbf{u}(t)-\mathbf{m}_\mathbf{u}^d)(\mathbf{u}(t)-\mathbf{m}_\mathbf{u}^d)^{\mathrm{T}})\end{cases}\quad\text{if not.} \tag{21}$$

In both cases, the set $(\boldsymbol{\psi}_k)_{1\leqslant k\leqslant n}$ constitutes an orthonormal basis of the Hilbert space $\mathbb{R}^d$.

4. Practical construction of POMs using a KL expansion

The construction of the POMs using the KL expansion requires, first, knowing the spatial covariance matrix function of the random displacement field under study with respect to the chosen averaging operator and, second, solving one of the eigenvalue problems (10), (14), or (19) associated to the chosen averaging operator. With linear random vibrations, when only the excitation is random, it is possible to characterize the covariance matrix function and thus to deduce some properties of the POMs; the computations can then be carried out and this will be illustrated in the next section. In the other cases the modeling of the POMs is carried out using experimental or numerical data (time-series and/or space-series data) solving the discretized eigenvalue problem.

We will assume in what follows that the displacements are measured at N spatial points, $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,\ldots,\mathbf{x}_N$, sampled in time M times, $t_m=m\tau$ with $m=1,\ldots,M$, where τ is the sampling period which must be greater than the correlation time, and for R independent realizations (or outcomes or independent random events) θ_r for $r=1,\ldots,R$. With these discretization parameters, we introduce the following notations:

- $\mathbf{u}(t_m,\mathbf{x}_n,\theta_r)$ denotes the observed or simulated value at $\mathbf{x}=\mathbf{x}_n$ and $t=t_m$ for the random event (or the realization) $\theta=\theta_r$,
- $\mathbf{U}_N(t_m,\theta_r)=(\mathbf{u}(t_m,\mathbf{x}_1,\theta_r)^{\mathrm{T}},\ldots,\mathbf{u}(t_m,\mathbf{x}_N,\theta_r)^{\mathrm{T}})$,

- $\mathbf{u}^{(m)}(.,\theta_r)$ is referred to as the snapshot at $t=t_m$ for the random event (or the realization) $\theta=\theta_r$ and $\mathbf{u}^{(m)}(\mathbf{x},\theta_r)=\mathbf{u}(t_m,\mathbf{x},\theta_r)$ denotes the snapshot value at the spatial point x.

4.1. Time-ergodic case

We assume here that the statistical properties of the random displacement field can be inferred from one single random event in the field (i.e., R is fixed at $R=1$) and that the ensemble average can be obtained from the time averaging or, at least, that the mean and covariance functions satisfy the following relations

$$E(\mathbf{u}(t,\mathbf{x}))=\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{u}(t,\mathbf{x},\theta_1)\,\mathrm{d}t \tag{22}$$

$$E(\mathbf{u}(t,\mathbf{x})\mathbf{u}(t+\tau,\mathbf{x}')^{\mathrm{T}})=\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{u}(t,\mathbf{x},\theta_1)\mathbf{u}(t+\tau,\mathbf{x}',\theta_1)^{\mathrm{T}}\,\mathrm{d}t. \tag{23}$$

The random field is time stationary and the approach presented in Section 3.1 can be used. In this case there are two practical methods available for determining the POMs: the direct and the snapshot method. Next they will be briefly presented and discussed.

4.1.1. Direct method

The random continuous field $\{\mathbf{u}(t,\mathbf{x})\}_{(t,,\mathbf{x})\in\mathscr{D}_T\times\mathscr{D}_\mathbf{x}}$ is approximated by the random discrete field $\{\mathbf{U}_N(t)\}_{t\in\mathscr{D}_T}$ and the KL theory described in Section 3.4 (Eqs. (19), (20)) can be used.

Introducing the centred displacement $\mathbf{V}_N(t_m)=\mathbf{U}_N(t_m,\theta_1)-(1/M)\sum_{k=1}^M\mathbf{U}_N(t_k,\theta_1)$ (for convenience, the argument θ_1 has been dropped) and drawing up the following $M\times dN$ ensemble matrix (where d is the dimension of the displacement flow):

$$\mathbf{V}=\begin{bmatrix}\mathbf{V}_N(t_1)\\ \mathbf{V}_N(t_2)\\ \vdots\\ \mathbf{V}_N(t_M)\end{bmatrix}, \tag{24}$$

the spatial correlation matrix with the dimension $dN\times dN$ can be written (using an ergodicity assumption) as follows

$$\mathbf{R}=\frac{1}{M}\mathbf{V}^{\mathrm{T}}\mathbf{V}. \tag{25}$$

The POMs are then approximated at the N spatial points, $\mathbf{x}_n$, by the eigenvectors of $\mathbf{R}$ (which are orthogonal due to its symmetry) and the eigenvalues will provide the POVs. The matrix dimensions obviously depend on the number of sampling points, N.

4.1.2. Snapshot method

This method was first introduced in Ref. [24]. It was based on the fact that, due to the assumed ergodicity, the spatial covariance matrix function can be expressed as

$$\mathbf{C}_u(0,\mathbf{x},\mathbf{x}')=\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\mathbf{v}^{(m)}(\mathbf{x})\mathbf{v}^{(m)}(\mathbf{x}')^{\mathrm{T}}, \tag{26}$$

where $\mathbf{v}^{(m)}(\mathbf{x})=\mathbf{u}^{(m)}(\mathbf{x},\theta_1)-(1/M)\sum_{k=1}^M\mathbf{u}^{(m)}(\mathbf{x},\theta_1)$ (for convenience, the argument θ_1 has been dropped). However, in practice, one would have to deal with a finite number of snapshots. This would affect the kernel $\mathbf{C}_u(0,\mathbf{x},\mathbf{x}')$ in Eq. (26), which could therefore have eigenfunctions which are linear combination of the snapshots [23,24]:

$$\boldsymbol{\psi}_k(\mathbf{x})=\sum_{m=1}^{M}A_{km}\mathbf{v}^{(m)}(\mathbf{x}), \tag{27}$$

where the coefficients A_{km} are still to be determined. Introducing Eqs. (26) and (27) into Eq. (15), these coefficients would provide solutions to the eigenvalue problem defined by

$$\mathbf{D}\mathbf{A}_k = \lambda_k \mathbf{A}_k \quad \text{with } D_{mn} = \frac{1}{M}\langle \mathbf{v}^{(m)}, \mathbf{v}^{(n)} \rangle_{\mathscr{D}_\mathbf{x}}. \tag{28}$$

To determine the POMs it is necessary, first, to compute the coefficients D_{mn} and, secondly, to perform the spectral decomposition of a matrix the dimensions of which depend on the number of snapshots M. The number of sampling points, N, enters the calculation only when the inner products are evaluated in Eq. (28).

It can thus be clearly seen that the direct method should generally be applied either to experimental data involving a rich time history obtained at a relatively small number of locations or to numerically generated data with moderate spatial resolution. Otherwise, in the case of multidimensional simulated flows with high spatial resolutions, the snapshot method is preferable.

4.2. Non-ergodic case

When the random displacement field is not ergodic we need to perform more than one realization to estimate the covariance matrix function. An estimator of the covariance matrix function can be obtained by averaging the data from several (independent) realizations. The direct and snapshot methods can be extended to this context using the averaging operator introduced in Section 3.2.

4.2.1. Direct method

As previously, the random continuous field $\{\mathbf{u}(t,\mathbf{x})\}_{(t,\mathbf{x})\in\mathscr{D}_T\times\mathscr{D}_\mathbf{x}}$ is approximated by the random discrete field $\{\mathbf{U}^N(t)\}_{t\in\mathscr{D}_T}$ and the KL theory as described in Section 3.4 (Eqs. (19), (21)) can be used.

Introducing the centred displacement $\mathbf{V}_N(t_m,\theta_r) = \mathbf{U}_N(t_m,\theta_r) - (1/R)\sum_{s=1}^{R}\mathbf{U}_N(t_m,\theta_s)$ and setting up the following $(MR)\times dN$ ensemble matrix gives:

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{N,M}(\theta_1) \\ \mathbf{V}_{N,M}(\theta_2) \\ \vdots \\ \mathbf{V}_{N,M}(\theta_R) \end{bmatrix} \quad \text{with } \mathbf{V}_{N,M}(\theta_r) = \begin{bmatrix} \mathbf{v}_N(t_1,\theta_r) \\ \mathbf{v}_N(t_2,\theta_r) \\ \vdots \\ \mathbf{v}_N(t_M,\theta_r) \end{bmatrix}, \tag{29}$$

and the spatial covariance matrix having the dimensions $dN\times dN$ can be written

$$\mathbf{R} = \frac{1}{MR}\mathbf{V}^{\mathrm{T}}\mathbf{V}. \tag{30}$$

The POMs are then approximated at the N spatial points, $\mathbf{x}_n$, by the eigenvectors of $\mathbf{R}$ (which are orthogonal due to the symmetry of $\mathbf{R}$) and the eigenvalues will provide the POVs. Eqs. (29) and (30) are an extension of Eqs. (24) and (25) taking the R realizations into account. The dimensions of the resulting matrix $\mathbf{R}$ do not depend on the number of realizations R. They are the same as those of the associated matrix in the ergodic case.

4.2.2. Snapshot method

This method is based on the fact that the covariance matrix function of $\{\{\mathbf{u}(\mathbf{x})\}\}_{\mathbf{x}\in\mathscr{D}_\mathbf{x}}$ can be expressed as follows

$$\mathbb{C}_u(\mathbf{x},\mathbf{x}') = \lim_{M,R\to\infty}\frac{1}{MR}\sum_{m=1}^{M}\sum_{r=1}^{R}\mathbf{v}^{(m,r)}(\mathbf{x})\mathbf{v}^{(m,r)}(\mathbf{x}')^{\mathrm{T}}, \tag{31}$$

where $\mathbf{v}^{(m,r)}(\mathbf{x}) = \mathbf{u}^{(m)}(\mathbf{x},\theta_r) - (1/R)\sum_{k=1}^{R}\mathbf{u}^{(m)}(\mathbf{x},\theta_k)$. However, in practice, one needs to deal with a finite number of snapshots. This could affect the kernel $\mathbb{C}_u(\mathbf{x},\mathbf{x}')$ in Eq. (31) and the eigenfunctions would therefore

be linear combinations of the snapshots:

$$\boldsymbol{\psi}_k(\mathbf{x}) = \sum_{m=1}^{M} \sum_{r=1}^{R} A_{kmr} \mathbf{v}^{(m,r)}(\mathbf{x}, \theta_r), \tag{32}$$

where the coefficients A_{kmr} are still to be determined. Introducing Eqs. (31) and (32) in to Eq. (15) these coefficients would provide solutions to the eigenvalue problem defined by

$$\mathbf{D}\mathbf{A}_k = \lambda_k \mathbf{A}_k \quad \text{with } D_{m_1 r_1 m_2 r_2} = \frac{1}{MR} \langle \mathbf{v}^{(m_1,r_1)}, \mathbf{v}^{(m_2,r_2)} \rangle_{\mathscr{D}_\mathbf{x}}. \tag{33}$$

Eqs. (32) and (33) are an extension of Eqs. (27) and (28)) taking the R realizations into account. The dimensions of matrix $\mathbf{D}$ are now equal to MR. This value is generally greater than the dimensions of the associated matrix in the direct method. This approach therefore seems to be less attractive than the direct method.

5. Relation between POMs using KL expansion and LNMs

5.1. Discrete linear case

Consider a discrete mechanical system with d degrees of freedom. Let $\mathbf{U}(t)$ be the displacement vector. We assume that $\mathbf{U}(t)$ satisfies the initial-value problem

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{F}(t), \quad t \in \mathscr{D}_T, \tag{34}$$

$$\mathbf{U}(0) = \mathbf{U}_0, \quad \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0, \tag{35}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric square matrices with dimensions $d \times d$, the vectors $\mathbf{U}_0$ and $\dot{\mathbf{U}}_0$ define the initial conditions of the motion, and $\{\mathbf{F}(t)\}_{t\in\mathscr{D}_T}$ is a random vector process.

The LNM are classically defined from the free responses of the associated undamped system as

$$\mathbf{K}\boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2,$$

where $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1 \cdots \boldsymbol{\Phi}_i \cdots \boldsymbol{\Phi}_n]$ denotes the modal matrix with the normalization condition $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}$ which implies that $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Omega}^2 = \mathrm{diag}(\omega_i^2)$; ω_i^2 and $\boldsymbol{\Phi}_i$ denote the squared resonance frequencies and the associated normal-mode vectors.

If the damping is not proportional, the modes are complex and this topic will be addressed in a future study. Here we focus on Eqs. (34) and (35) with proportional damping. Note that in this case the matrix $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi}$ is also diagonal. In this section the aim is to establish when the POMs defined in Section 3 (which were based on forced responses) can be used to determine the LNM. This part of the study which is in line with the results previously presented in Refs. [15,19,25] will be restricted to the case where the excitation is a white-noise random process with zero mean, and the method used to characterize covariance matrix function, recalled in Appendix A, will be used.

5.1.1. Case I: $\mathbf{M} = m\mathbf{I}$

We assume here that $\mathbf{M} = m\mathbf{I}$, $m > 0$. Using the normal mode vectors as the basis of modeling, the modal-displacement vector $\mathbf{Q}(t)$ defined by

$$\mathbf{U}(t) = \boldsymbol{\Phi}\mathbf{Q}(t) = \sum_{i=1}^{d} \boldsymbol{\Phi}_i Q_i(t) \tag{36}$$

satisfy the following second-order differential equation

$$\ddot{\mathbf{Q}}(t) + \boldsymbol{\Theta}\dot{\mathbf{Q}}(t) + \boldsymbol{\Omega}^2\mathbf{Q}(t) = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{F}(t), \tag{37}$$

where $\boldsymbol{\Theta} = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi} = \mathrm{diag}(2\tau_i\omega_i)$ is diagonal.

The evolution of the covariance matrix, $\mathbf{C}_{\mathbb{Q}}(t) = E(\mathbb{Q}(t)\mathbb{Q}^{\mathrm{T}}(t))$, of $\mathbb{Q}(t) = (\mathbf{Q}^{\mathrm{T}}(t), \dot{\mathbf{Q}}^{\mathrm{T}}(t))^{\mathrm{T}}$ is given by (see Appendix A)

$$\dot{\mathbf{C}}_{\mathbb{Q}}(t) = \mathbf{A}_{\mathbb{Q}}\mathbf{C}_{\mathbb{Q}}(t) + \mathbf{C}_{\mathbb{Q}}(t)\mathbf{A}_{\mathbb{Q}}^{\mathrm{T}} + \mathbf{D}_{\mathbb{Q}}, \tag{38}$$

$$\mathbf{C}_{\mathbb{Q}}(0) = \mathbf{C}_{\mathbb{Q}_0}, \tag{39}$$

where

$$\mathbf{A}_{\mathbb{Q}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Theta} & -\mathbf{\Omega}^2 \end{pmatrix}, \quad \mathbf{D}_{\mathbb{Q}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}^{\mathrm{T}}\mathbf{S}_F\mathbf{\Phi} \end{pmatrix},$$

$\mathbf{C}_{\mathbb{Q}_0}$ is easily deduced from $\mathbf{C}_{\mathbb{U}_0}$, and the covariance matrix, $\hat{\boldsymbol{C}}_{\mathbb{Q}}$, of the stationary response is given by

$$0 = \mathbf{A}_{\mathbb{Q}}\hat{\boldsymbol{C}}_{\mathbb{Q}} + \hat{\boldsymbol{C}}_{\mathbb{Q}}\mathbf{A}_{\mathbb{Q}}^{\mathrm{T}} + \mathbf{D}_{\mathbb{Q}}. \tag{40}$$

If the matrix $\mathbf{\Phi}^{\mathrm{T}}\mathbf{S}_F\mathbf{\Phi}$ is diagonal (i.e., when the modal-excitation terms $\mathbf{\Phi}_i^{\mathrm{T}}\mathbf{F}(t)$ in Eq. (37) are uncorrelated), it is easy to establish from Eq. (40) that the stationary covariance matrix $\hat{\boldsymbol{C}}_{\mathbb{Q}}$ and the stationary covariance matrix $\hat{\boldsymbol{C}}_{\mathbf{Q}}$ of the modal displacement are diagonal. Now, recalling the change of variables (36), we obtain the following relation:

$$\hat{\boldsymbol{C}}_U = \mathbf{\Phi}\hat{\boldsymbol{C}}_{\mathbf{Q}}\mathbf{\Phi}^{\mathrm{T}} \tag{41}$$

from which we can deduce (recalling $\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi} = m^{-1}\boldsymbol{I}$) that the results of the POD performed using the KL expansion of the stationary response $\mathbf{U}(t)$ (see Section 3.4, Eqs. (19) and (20)) agree with those of the modal expansion (36).

If the matrices $\mathbf{\Phi}^{\mathrm{T}}\mathbf{S}_F\mathbf{\Phi}$ and $\mathbf{C}_{\mathbb{Q}_0}$ are diagonal, the same property holds for the POD obtained using the KL expansion of the transient response over $[0, T]$ with arbitrary T (see Section 3.4, Eqs. (19) and (21)). From Eq. (38), it can be seen that the covariance matrix function, $\mathbf{C}_{\mathbb{Q}}(t)$, is a 2×2 block matrix with diagonal blocks of the same size. For all $t \in \mathscr{D}_T$, the matrix $\boldsymbol{C}_{\mathbf{Q}}(t)$ is diagonal, and integrating the relation $\boldsymbol{C}_{\mathbf{U}}(t) = \mathbf{\Phi}^{\mathrm{T}}\boldsymbol{C}_{\mathbf{Q}}(t)\mathbf{\Phi}$ over $[0, T]$ gives

$$\frac{1}{T}\int_0^T \boldsymbol{C}_{\mathbf{U}}(t)\,\mathrm{d}t = \mathbf{\Phi}\frac{1}{T}\int_0^T \boldsymbol{C}_{\mathbf{Q}}(t)\,\mathrm{d}t\,\mathbf{\Phi}^{\mathrm{T}} \tag{42}$$

from which the above result is deduced.

As the modes of a damped linear system coincide with the LNM if the modal matrix diagonalizes the damping matrix, the POMs coincide with the LNM if the modal matrix diagonalizes the covariance matrix of the excitation.

5.1.2. Case II: unspecified mass matrix

Introducing the square root, $\mathbf{M}^{1/2}$, of the matrix $\mathbf{M}$ (i.e. $\mathbf{M} = \mathbf{M}^{1/2}\mathbf{M}^{1/2}$) and using the change of variable

$$\mathbf{V} = \mathbf{M}^{1/2}\mathbf{U} \tag{43}$$

the equation of motion (34) reads

$$\ddot{\mathbf{V}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{V}}(t) + \tilde{\mathbf{K}}\mathbf{V}(t) = \mathbf{M}^{-1/2}\mathbf{F}(t), \tag{44}$$

where the new damping and stiffness matrices, $\tilde{\mathbf{C}} = \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}$ and $\tilde{\mathbf{K}} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$, are still symmetric matrices. The previous results can now be applied to the new variable $\mathbf{V}$.

Let $\mathbf{\Psi}_i$ (for $i = 1, \ldots, d$) be the linear normal mode vectors (with the normalization condition $\mathbf{\Psi}^{\mathrm{T}}\mathbf{\Psi} = \mathbf{I}$, where $\mathbf{\Psi}$ denotes the modal matrix, i.e., $\mathbf{\Psi} = [\mathbf{\Psi}_1 \cdots \mathbf{\Psi}_i \cdots \mathbf{\Psi}_d]$) of system (44).

Following the result established in Section 5.1.1, if the matrix $\mathbf{\Psi}^{\mathrm{T}}\mathbf{M}^{-1/2}\mathbf{S}_F\mathbf{M}^{-1/2\mathrm{T}}\mathbf{\Psi}$ is diagonal, then the POMs obtained using the KL expansion of the stationary response $V(t)$ are in agreement with the LNM, $\mathbf{\Psi}_i$. Knowing the mass matrix, $\boldsymbol{M}$, it is possible to extract the LNM of the original system (34) from $\mathbf{\Psi}$ using the relation

$$\mathbf{\Phi} = \mathbf{M}^{-1/2}\mathbf{\Psi}. \tag{45}$$

Hence, when $\mathbf{M} \neq m\mathbf{I}$, the POMs obtained using the KL expansion of the stationary response $\mathbf{U}(t)$ do not coincide with the modal expansion and the LNMs can only be obtained from the variable $\mathbf{V}$. Note that the condition can be written as follows using the LNM

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{M}^{-1/2}\mathbf{S}_F\mathbf{M}^{-1/2\mathrm{T}}\mathbf{\Psi} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}_F\mathbf{\Phi}. \tag{46}$$

A similar result can be obtained for the transient response but this point will not be addressed here.

5.1.3. Influence of the correlation coefficient between modal excitation terms

As we have seen above the KL expansion can be used to obtain the LNM if the modal excitation terms $\mathbf{\Phi}_i^{\mathrm{T}}\mathbf{F}(t)$ are uncorrelated. In this section we will discuss the influence of the correlation coefficient between modal excitation terms.

Let us take a 2-degree-of-freedom (dof) linear system (34) and (35) with proportional damping and identity mass matrix. We assume that the matrix $\mathbf{\Phi}^{\mathrm{T}}\mathbf{S}_F\mathbf{\Phi}$ is not diagonal. Let

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{S}_F\mathbf{\Phi} = \begin{pmatrix} \sigma_{11} & \rho\sqrt{\sigma_{11}\sigma_{22}} \\ \rho\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{11} \end{pmatrix}, \tag{47}$$

where σ_{11} and σ_{22} denote the modal input level and ρ the associated correlation coefficient. Solving Eq. (40), the stationary covariance matrix $\hat{\boldsymbol{C}}_{\mathbf{Q}}$ has components

$$\hat{C}_{Q_{11}} = \frac{\sigma_{11}}{4\tau_1\omega_1^3}, \quad \hat{C}_{Q_{22}} = \frac{\sigma_{22}}{4\tau_2\omega_2^3}, \quad \hat{C}_{Q_{12}} = \rho_{Q_{12}}\sqrt{\hat{C}_{Q_{11}}\hat{C}_{Q_{22}}} \tag{48}$$

with

$$\rho_{Q_{12}} = \rho\frac{8\tau_1^2 r_\omega\sqrt{r_\tau r_\omega}(1 + r_\tau r_\omega)}{(1 - r_\omega^2)^2 + 4\tau_1^2(1 + r_\tau r_\omega)(r_\tau + r_\omega)r_\omega}, \quad r_\tau = \frac{\tau_2}{\tau_1}, \quad r_\omega = \frac{\omega_2}{\omega_1}, \tag{49}$$

where ω_i and τ_i denote the resonance frequencies and the associated damping ratios (see Eq. (37)) of the two modal components under consideration. Introducing the ratio $r_\sigma = \sigma_{22}/\sigma_{11}$, the stationary covariance matrix takes the form

$$\hat{\boldsymbol{C}}_{\mathbf{Q}} = \frac{\sigma_{11}}{4\tau_1\omega_1^3}\begin{pmatrix} 1 & \rho_{Q_{12}}\sqrt{\dfrac{r_\sigma}{r_\tau r_\omega^3}} \\ \rho_{Q_{12}}\sqrt{\dfrac{r_\sigma}{r_\tau r_\omega^3}} & \dfrac{r_\sigma}{r_\tau r_\omega^3} \end{pmatrix} \tag{50}$$

showing that its eigenvectors depend only on the modal damping (τ_1, τ_2), modal frequency ratio r_ω, modal input level ratio r_σ, and the correlation coefficient ρ. Note that the eigenvectors do not depend on the absolute values of the modal frequencies.

Figs. 1 and 2 show the Euclidian norm of the error vector between the canonical vector $e_1 = (1, 0)^{\mathrm{T}}$ and the normalized eigenvector of $\hat{\boldsymbol{C}}_{\mathbf{Q}}$, plotted versus the correlation coefficient ρ for $\tau = \tau_1 = \tau_2 = 0.01$ and $\tau = \tau_1 = \tau_2 = 0.1$, respectively. The difference between the KLM and the LNM increases with ρ and decreases as the modal frequency ratio increases. The dotted lines show the error vector between the canonical vector $e_1 = (1, 0)^{\mathrm{T}}$ and the normalized eigenvector of $(1/T)\int_0^T \boldsymbol{C}_{\mathbf{Q}}(t)\,\mathrm{d}t$ where the matrix function $\boldsymbol{C}_{\mathbf{Q}}(t)$ has been obtained solving Eq. (38) over $[0, T]$ numerically, with $T = 200$ and $T = 100$, and with $\tau = 0.1$ and $\tau = 0.01$, respectively. The influence of T will be discussed in the following section.

5.1.4. Influence of T

The KL expansion of a non-stationary process has to be built using the averaging operator $\mathbb{E}(.)$ and, in this case, the result depends on the time duration T of the observation. For mechanical systems as defined by Eqs. (34) and (35) when T tends to ∞, the KL expansion coincides with the KL expansion given by the stationary response (see Appendix A).

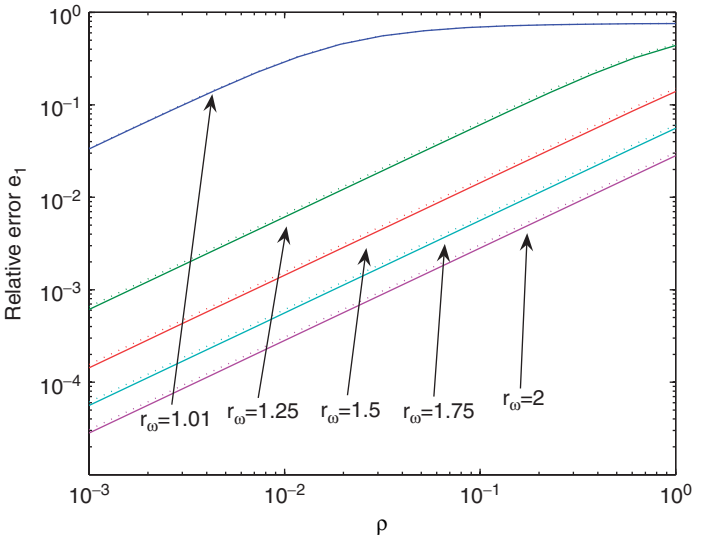

Fig. 1. Euclidian norm of the error vector between the canonical vector $e_1 = (1,0)^{\mathrm{T}}$ and the normalized eigenvector of $\hat{\boldsymbol{C}}_X$ versus the correlation coefficient ρ with $\tau = 0.1$.

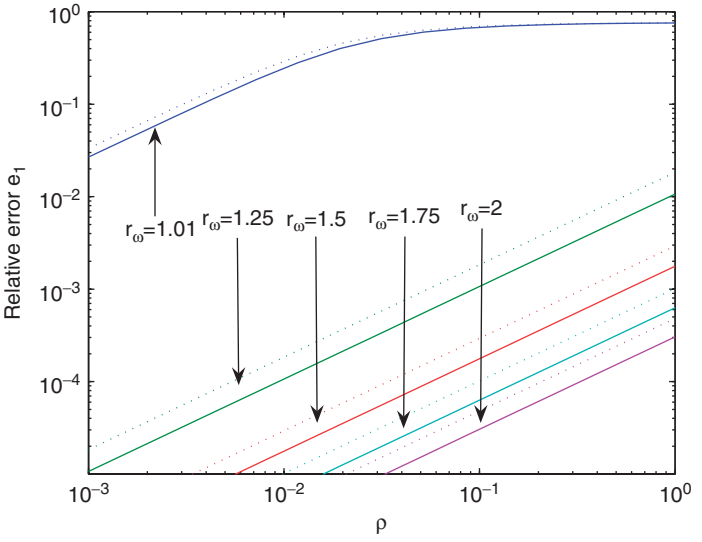

Fig. 2. Euclidian norm of the error vector between the canonical vector $e_1 = (1,0)^{\mathrm{T}}$ and the normalized eigenvector of $\hat{\boldsymbol{C}}_X$ versus the correlation coefficient ρ with $\tau = 0.01$.

Considering the same example as in Section 5.1.3, we discuss the influence of T on the KL expansion. The time constant of the mechanical system is used as a time unit. Using the same notations as in Section 5.1.3, the time constant is defined by $T_c = \max(1/\tau_1\omega_1, 1/\tau_2\omega_2)$.

Figs. 3 and 4 show the Euclidian norm of the error vector between the canonical vector $e_1 = (1,0)^{\mathrm{T}}$ and the normalized eigenvector of the covariance matrix

$$\hat{\boldsymbol{C}}_{\mathbf{Q}}(T) = \frac{1}{T}\int_0^T \boldsymbol{C}_{\mathbf{Q}}(t)\,\mathrm{d}t,$$

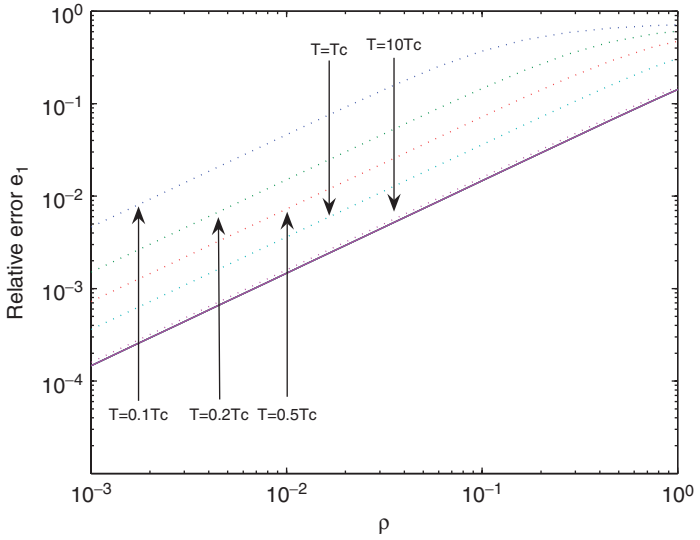

Fig. 3. Euclidian norm of the error vector between the canonical vector $e_1 = (1,0)^{\mathrm{T}}$ and the normalized eigenvector of $\mathbb{C}_{\mathbf{Q}}(T)$ versus the correlation coefficient ρ (dashed lines) with $\tau = 0.1$ and $\omega_2 = 1.5\omega_1$.

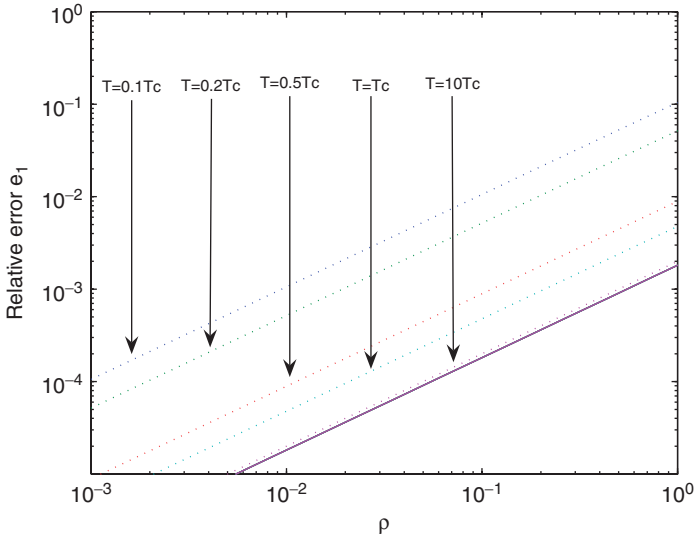

Fig. 4. Euclidian norm of the error vector between the canonical vector $e_1 = (1,0)^{\mathrm{T}}$ and the normalized eigenvector of $\mathbb{C}_{\mathbf{Q}}(T)$ versus the correlation coefficient ρ (dashed lines) with $\tau = 0.01$ and $\omega_2 = 1.5\omega_1$.

plotted versus the correlation coefficient ρ for $\tau = \tau_1 = \tau_2 = 0.01$ and $\tau = \tau_1 = \tau_2 = 0.1$, respectively. As expected, for T large ($T \geqslant 10T_c$, the KL expansion given by $\hat{\boldsymbol{C}}_{\mathbf{Q}}(T)$ is close to the stationary case (continuous line)).

5.2. Continuous linear case

Here we deal with the case of a beam. Let $\varphi_i(x)$ be the modal functions (where $\int_0^L \varphi_i(x)\varphi_j(x)\,\mathrm{d}x = \delta_{ij}$ and L denotes the length of the beam). The displacements of the beam can be expanded into a

truncated series

$$u(t,x)=\sum_{i=1}^{p}\varphi_i(x)Q_i(t), \tag{51}$$

where the modal components can be modelled as follows:

$$\ddot{\mathbf{Q}}(t)+\mathbf{\Theta}\dot{\mathbf{Q}}(t)+\mathbf{\Omega}^2\mathbf{Q}(t)=\mathbf{G}(t), \tag{52}$$

where $\mathbf{\Omega}^2=\mathrm{diag}(\omega_i^2)$ and $\mathbf{\Theta}=\mathrm{diag}(2\tau_i\omega_i)$. The ω_i denote the modal frequencies and τ_i the associated modal damping. The components of the modal excitation vector $\mathbf{G}(t)$ can be related to the physical excitation $F(t,x)$ by

$$G_i(t)=\int_0^L\varphi_i(x)F(t,x)\,\mathrm{d}x.$$

In our case, $F(t,x)$ is a random process and we will assume that $G(t)$ is a white-noise random process.

Using the same arguments as in Section 5.1.1, it is possible to show that if the covariance matrix of the modal excitation, $\mathbf{G}(t)$, is diagonal then the PODs obtained using the KL expansion will coincide with the modal expansion (51). Unfortunately this is not generally the case, and the KL theory cannot therefore be used to obtain the modal functions.

As observed in Ref. [19], another source of perturbation is introduced by the spatial sampling process. Let $x_k=k\Delta x$ for $k=1,\ldots,N$ with $\Delta x=L/N$. The covariance matrix of the discrete response $\mathbf{U}(t)=(u(t,x_k))$ is related to the covariance matrix of the modal components $\mathbf{Q}(t)$ by $\mathbf{C}_\mathbf{U}=\mathbf{\Gamma}\mathbf{C}_\mathbf{Q}\mathbf{\Gamma}^\mathrm{T}$ where $\mathbf{\Gamma}$ is a $N\times q$ matrix with components $\Gamma_{ki}=(\varphi_i(x_k))$.

If $\mathbf{\Gamma}^\mathrm{T}\mathbf{\Gamma}=\boldsymbol{I}$, then $\mathbf{C}_\mathbf{U}\mathbf{\Gamma}=\mathbf{\Gamma}\mathbf{C}_\mathbf{Q}$ and the eigenvectors of $\mathbf{C}_\mathbf{Q}$ are given by the eigenvectors of $\mathbf{C}_\mathbf{U}$ multiplied by the matrix $\mathbf{C}$. Note that the qualitative errors mentioned in Section 5.1.3 still occur here (if the weight matrix $\mathbf{\Gamma}^\mathrm{T}\mathbf{\Gamma}=\boldsymbol{I}$). Unfortunately, in practice we never have $\mathbf{\Gamma}^\mathrm{T}\mathbf{\Gamma}=\boldsymbol{I}$, and hence the qualitative errors mentioned in Section 5.1.3 are not valid. We expect the difference between the POMs obtained using KL expansion and the LNM to increase as N decreases.

In Figs. 5 and 6, the exact first, second, fifth and sixth modal functions of a clamped–free beam are compared with the associated POMs obtained using the KL expansion of the stationary response and the transient response with the initial condition $\mathbf{C}_{\mathbb{Q}_0}=\mathbf{0}$. A localized excitation force $F(x,t)=\delta(x-x_f)f(t)$ where $f(t)$ is a scalar random process with covariance function $C_f(\tau)=E(f(t+\tau)f^\mathrm{T}(t))=S_f\delta(\tau)$ has been used. The parameter values are: $L=0.6$, $EI=1.4$, $\rho S=0.1620$, $p=10$, $\tau_i=\tau=0.01$, $x_f=0.05$ (all the modes were excited and the correlation coefficient between pairs of modal components were equal to 1), $S_f=1$, $T=1$ (which correspond to approximately four fundamental periods of the smaller resonance frequency). As was to be expected, it turned out that the POMs differ from the LNMs. This difference decreases as N increases ($N=10$ in Fig. 5 and $N=40$ in Fig. 6). Due to the values of the ratio between successive resonance frequencies, the difference also decreases as the order of the mode increases. The only differences observed between the POMs associated with the stationary response and the POMs associated with the transient response were between the first two modes.

6. Some comments on the nonlinear case

One rather interesting result was the difference between the POMs obtained using the KL expansion of the response of the nonlinear system and the POMs obtained using the KL expansion of the response of the equivalent linear system obtained using the method of statistical linearization as described in Ref. [27]. Let us consider the nonlinear system

$$\dot{\mathbf{Z}}(t)=\mathbf{G}(\mathbf{Z}(t))+\mathbf{F}(t) \tag{53}$$

with external random excitation. A suitable equivalent linear system relationship between $\mathbf{Z}(t)$ and $\mathbf{F}(t)$ can be written as follows

$$\dot{\mathbf{Z}}(t)=\mathbf{L}_\mathrm{eq}\mathbf{Z}(t)+\mathbf{F}(t), \tag{54}$$

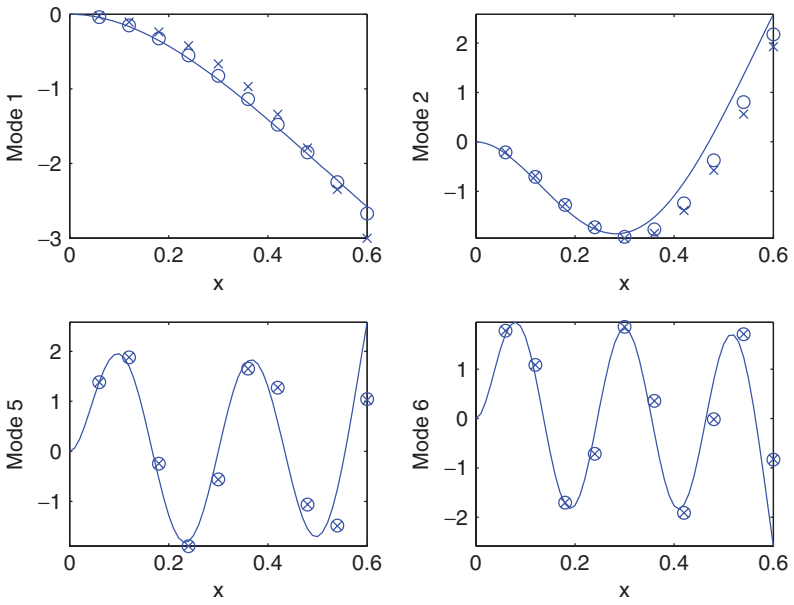

Fig. 5. Modal functions (solid line) of clamped–free beam and corresponding POMs obtained using KL expansion of the stationary response (○), and the transient response over $[0, T]$ (×) with $N = 10$.

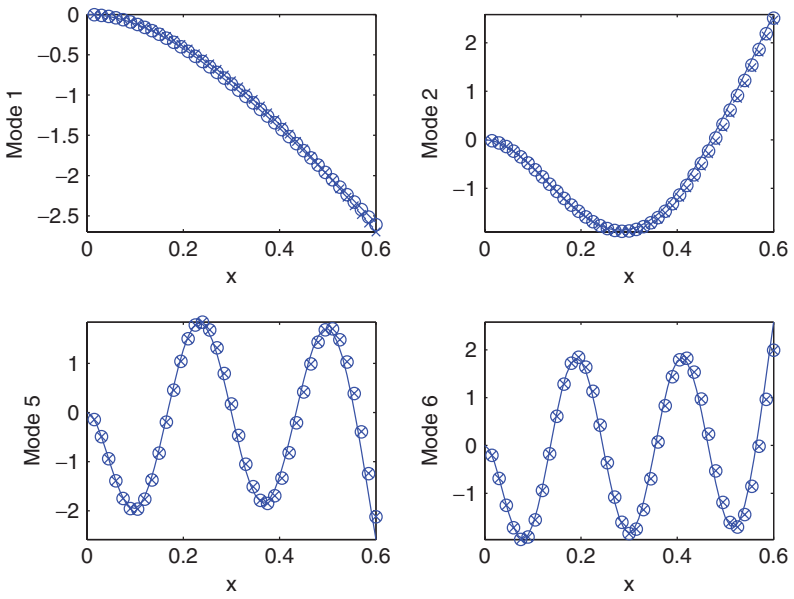

Fig. 6. Modal functions (solid line) of clamped–free beam and corresponding POMs obtained using KL expansion of the stationary response (○), and the transient response over $[0, T]$ (×) with $N = 40$.

where the matrix constant $\mathbf{L}_{\text{eq}}$ is determined by

$$\min_{\mathbf{L}} E(\|\mathbf{G}(\mathbf{Z}(t)) - \mathbf{L}\mathbf{Z}(t)\|^2). \tag{55}$$

For the nonlinear system (53), where there exists a stationary, ergodic probability measure, it can be shown [27] that the stationary covariance matrix of the nonlinear response (53) is identical to the stationary covariance matrix of the equivalent linear response (54).

Assuming the existence of stationary conditions, the POMs obtained using the KL expansion of the stationary response of the nonlinear system agree with the POMs obtained using the KL expansion of the stationary response of the equivalent linear system.

We will now study the case of transient (or non-stationary) responses. Let consider the nonlinear system

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{G}(\mathbf{U}(t)) = \mathbf{F}(t), \quad t \in [0, T], \tag{56}$$

$$\mathbf{U}(0) = \mathbf{U}_0, \quad \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0. \tag{57}$$

A suitable equivalent linear system relationship between $\mathbf{U}(t)$ and $\mathbf{F}(t)$ can be written as follows:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}_{\text{eq}}\mathbf{U}(t) = \mathbf{F}(t), \quad t \in [0, T], \tag{58}$$

where the constant matrix $\mathbf{K}_{\text{eq}}$ is determined by

$$\min_{\mathbf{K}} \mathbb{E}(\|\mathbf{G}(\mathbf{U}(.)) - \mathbf{K}\mathbf{U}(.)\|^2) \tag{59}$$

with $\mathbb{E}(.) = (1/T)\int_0^T E(.)\,\mathrm{d}t$. This criterion differs from the stationary one (55). It can be used to obtain an equivalent linear system with a constant matrix. This linearization method differs from that described in Ref. [28] in the case of non-stationary responses where the equivalent linear system was a time-varying linear system.

As in Ref. [28], the condition required to obtain optimum can be written as follows:

$$\mathbb{E}(\mathbf{U}(.)\mathbf{U}^{\mathrm{T}}(.))\mathbf{K}_{\text{eq}}^{\mathrm{T}} = [\mathbb{E}(G_1(\mathbf{U}(.))\mathbf{U}(.)) \cdots \mathbb{E}(G_d(\mathbf{U}(.))^{\mathrm{T}}\mathbf{U}(.))], \tag{60}$$

where $\mathbf{G}(\mathbf{U}) = (G_1(\mathbf{U})G_2(\mathbf{U}) \cdots G_d(\mathbf{U}))^{\mathrm{T}}$.

The question now is: do the POMs obtained using the KL expansion of the nonlinear transient response (56) agree with the POMs obtained using the KL expansion of the transient response of the equivalent linear system (58)?

First, the following comment should be made from simulation results. We consider the clamped–free beam where the free end is fixed to a cubic spring $(\lambda u(L, t)^3)$.

The displacement histories were obtained from excitation histories by solving Eq. (52) (including the nonlinear term) numerically using the Newmark method. The excitation histories were simulated using the procedure described in Ref. [26]. The same parameter values as in Section 5.2 were used with the nonlinear parameter value $\lambda = 10^6$ and the time discretization parameter value $\tau = 0.00025$ (giving $M = 4000$ and recalling that $T = 1$). $R = 1000$ realizations were computed. The POMs obtained using the KL expansion of the transient nonlinear response were computed using the direct method (see Section 4.2.1). The simulated data were also used to estimate $\mathbf{K}_{\text{eq}}$ solving Eq. (60) and the POMs obtained using the KL expansion of the transient response of the equivalent linear system (58) were computed solving the associated Eq. (63) of the covariance matrix function.

In Figs. 7–10, the first, second, fifth, and sixth modes, respectively, of the underlying linear clamped–free beam, the corresponding POMs obtained using the KL expansions of the transient responses of the nonlinear system and those of the equivalent linear system are compared. First of all, we can observe that the POMs obtained with the two systems (the nonlinear and the equivalent linear system) are very similar. The result which holds true when we are looking for the stationary responses using the averaging operation $(E(.))$ seems to be reasonably true in the case of transient response using the averaging operation $(\mathbb{E}(.))$. Of course, the proof of this concordance still needs to be established theoretically. As mentioned above, the nonlinear effect appears to be more significant in the first two modes, and to be less pronounced in the higher-order modes. We have also plotted, in these figures, several eigenvectors obtained from single realizations of displacement history. These modes were computed using the direct method described in Section 4.2.1 with the parameter value $R = 1$ or, equivalently, with the direct method described in Section 4.1.1 for various displacement history data. The eigenvectors obviously differ from the POMs as well as from the LNMs. Depending on the realization, the difference with respect to the POMs can be significant (see for example Fig. 9).

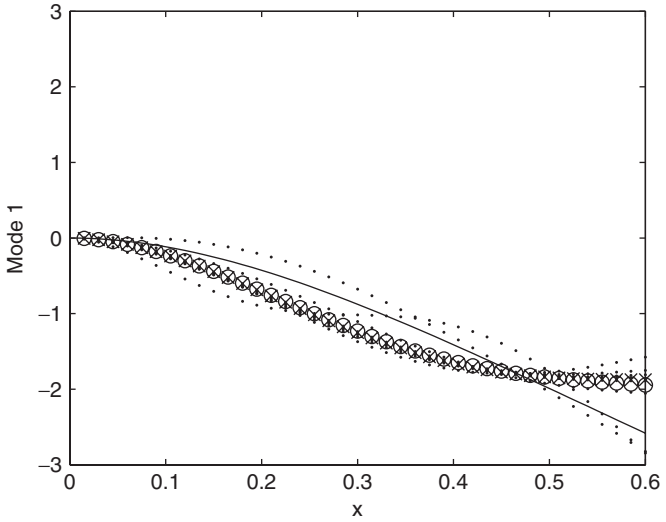

Fig. 7. The first mode (solid line) of a linear clamped–free beam, the corresponding POMs obtained using KL expansion of the transient nonlinear response (○) and the transient response of the equivalent linear system over $[0, T]$ (×), and that of the nonlinear beam obtained with several sampled trajectories (·) with $N = 40$.

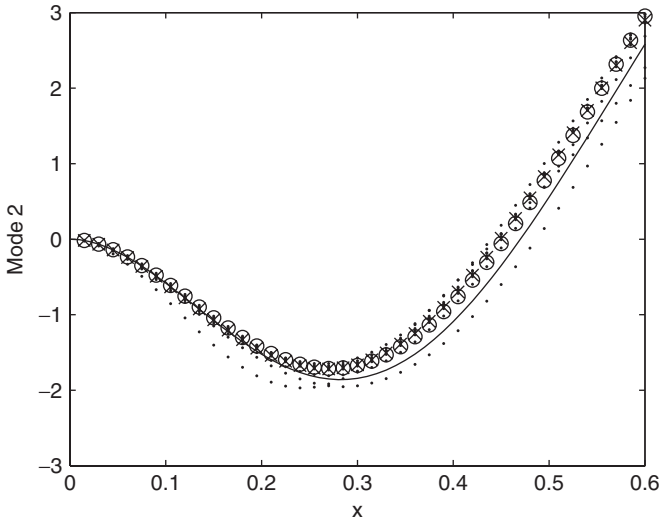

Fig. 8. The second mode (solid line) of a linear clamped–free beam, the corresponding POMs obtained using KL expansion of the transient nonlinear response (○) and the transient response of the equivalent linear system over $[0, T]$ (×), and the corresponding POM modes of the nonlinear beam obtained with several sampled trajectories.

7. Conclusion

In this present study we show how the KL expansion can be used to obtain a POD of the randomly excited vibrating system responses in separated variables (time and space) form including transient and stationary response cases. An averaging operator involving time and ensemble averages has been used to develop the KL theory. The results obtained with this approach are in line with those obtained using the classical POD method in the case of deterministic or ergodic random excitation.

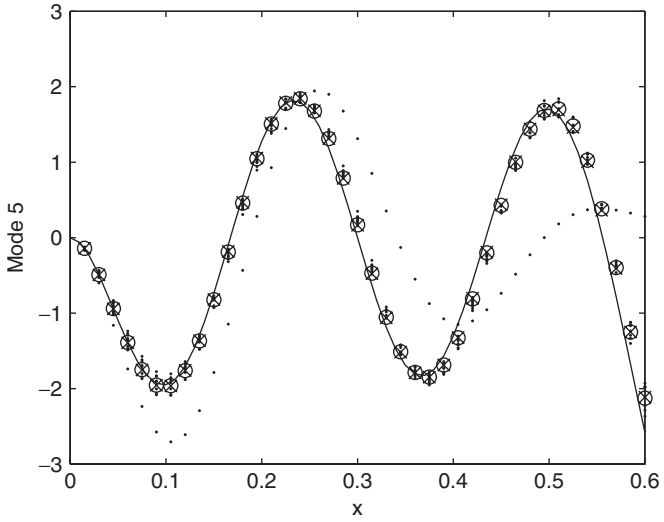

Fig. 9. The fifth mode (solid line) of a linear clamped–free beam, the corresponding POMs obtained using KL expansion of the transient nonlinear response (○) and the transient response of the equivalent linear system over $[0, T]$ (×), and the corresponding POM modes of the nonlinear beam obtained with several sampled trajectories.

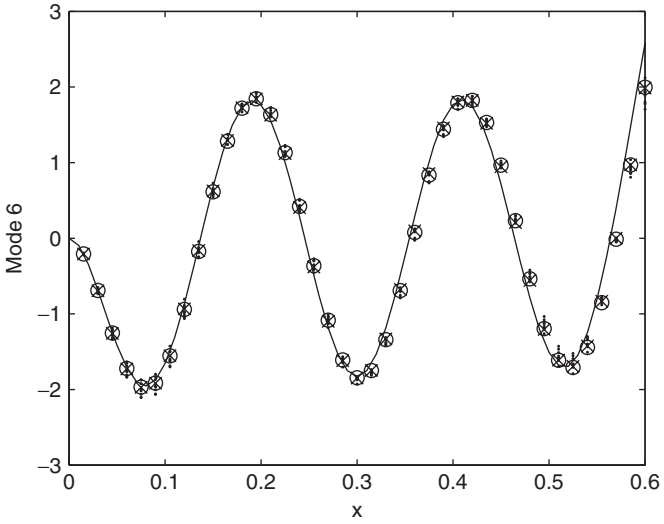

Fig. 10. The sixth mode (solid line) of a linear clamped–free beam, the corresponding POMs obtained using KL expansion of the transient nonlinear response (○) and the transient response of the equivalent linear system over $[0, T]$ (×), and the corresponding POM modes of the nonlinear beam obtained with several sampled trajectories.

The classical direct and snapshot methods have been extended here to perform the expansion based on experimental data. The snapshot method seems to be less efficient in the non-ergodic than in the ergodic case.

The POMs are interpreted here in the case of linear and nonlinear vibrating systems subjected to white-noise excitation in terms of normal modes. As previously mentioned by several authors, a mass condition is necessary to make the POMs and the LNMs agree. Moreover, in damped linear systems, the POMs and LNMs agree if the modal matrix diagonalizes both the damping matrix and the covariance excitation matrix. In the nonlinear case, the POMs are related to the POMs of the equivalent linear system obtained using the statistical linearization method.

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Appendix A. Derivation of the covariance matrix evolution

Consider the linear d-degrees-of-freedom equation of motion (34) and (35).

Let $\mathbb{U}(t) = (\mathbf{U}^{\mathrm{T}}(t), \dot{\mathbf{U}}^{\mathrm{T}}(t))^{\mathrm{T}}$. Under the assumption that the initial conditions are deterministic (or at least uncorrelated with the excitation), it can be established that the covariance matrix function, $\mathbf{C}_{\mathbb{U}}(t) = E((\mathbb{U}(t) - \mathbf{m}_{\mathbb{U}}(t))(\mathbb{U}(t) - \mathbf{m}_{\mathbb{U}}(t))^{\mathrm{T}})$ satisfies the following differential equation:

$$\dot{\mathbf{C}}_{\mathbb{U}}(t) = \mathbf{A}_{\mathbb{U}}\mathbf{C}_{\mathbb{U}}(t) + \mathbf{C}_{\mathbb{U}}(t)\mathbf{A}_{\mathbb{U}}^{\mathrm{T}} + \mathbf{Z}(t) + \mathbf{Z}^{\mathrm{T}}(t), \quad t \in \mathscr{D}_T, \tag{61}$$

$$\mathbf{C}_{\mathbb{U}}(0) = \mathbf{C}_{\mathbb{U}_0}, \tag{62}$$

where $\mathbf{C}_{\mathbb{U}_0}$ is the covariance matrix of the random vector $\mathbb{U}(0) = (\mathbf{U}_0^{\mathrm{T}}, \dot{\mathbf{U}}_0^{\mathrm{T}})^{\mathrm{T}}$,

$$\mathbf{A}_{\mathbb{U}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix},$$

$$\mathbf{Z}(t) = \int_0^t \mathrm{e}^{\mathbf{A}_{\mathbb{U}}(t-\tau)} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}\mathbf{C}_F(t,\tau)\mathbf{M}^{-1^T} \end{pmatrix} \mathrm{d}\tau,$$

where $\mathbf{C}_{\mathbf{F}}(t_1, t_2) = E((\mathbf{F}(t_1) - \mathbf{m}_{\mathbf{F}}(t_1))(\mathbf{F}(t_2) - \mathbf{m}_{\mathbf{F}}(t_2))^{\mathrm{T}})$ denotes the covariance matrix function of the random vector process $\{\mathbf{F}(t)\}_{t\in\mathscr{D}_T}$.

In addition, if $\{\mathbf{F}(t)\}_{t\in\mathscr{D}_T}$ is a zero-mean white-noise excitation process that is $\mathbf{C}_F(\tau) = \mathbb{E}(\mathbf{F}(t+\tau)\mathbf{F}^{\mathrm{T}}(t)) = \mathbf{S}_F\delta(\tau)$ where $\mathbf{S}_F$ is a constant symmetric matrix then Eq. (61) reduces to

$$\dot{\mathbf{C}}_{\mathbb{U}}(t) = \mathbf{A}_{\mathbb{U}}\mathbf{C}_{\mathbb{U}}(t) + \mathbf{C}_{\mathbb{U}}(t)\mathbf{A}_{\mathbb{U}}^{\mathrm{T}} + \mathbf{D}_{\mathbb{U}}, \quad \mathbf{D}_{\mathbb{U}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}\mathbf{S}_F\mathbf{M}^{-1^{\mathrm{T}}} \end{pmatrix}. \tag{63}$$

When A is a stability matrix (this is the case when the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric and positive definite), the matrix function $\mathbf{C}_{\mathbb{U}}(t)$ tends to a symmetrical matrix $\hat{\mathbf{C}}_{\mathbb{U}}$ which solves the following Lyapunov equation:

$$\mathbf{0} = \mathbf{A}_{\mathbb{U}}\hat{\mathbf{C}}_{\mathbb{U}} + \hat{\mathbf{C}}_{\mathbb{U}}\mathbf{A}_{\mathbb{U}}^{\mathrm{T}} + \mathbf{D}_{\mathbb{U}}. \tag{64}$$

Recalling that the covariance matrix $\mathbf{C}_U(t)$ of the response $U(t)$ is equal to the first block with dimension $n \times n$ of the matrix $\mathbf{C}_{\mathbb{U}}(t)$, we can now use Eqs. (63) and (64) to analyse the behaviour of the KL decomposition from transient to stationary responses of system (34) and (35). The transient response is characterized by the differential equation (63) whereas the stationary response is characterized by the algebraic equation (64).

To analyse the transient response the averaging operator $\mathbb{E}(.)$ has to be used. It can be shown, from Eq. (63), that the covariance matrix function

$$\mathbb{C}_{\mathbb{U}}(T) = \frac{1}{T}\int_0^T \mathbf{C}_{\mathbb{U}}(t)\,\mathrm{d}t \tag{65}$$

satisfies

$$\mathbb{C}_{\mathbb{U}}(T) \to \hat{\mathbf{C}}_{\mathbb{U}} \quad \text{when } T \to \infty. \tag{66}$$

Hence, for large T, the KL decomposition based on the averaging operator $\mathbb{E}(.)$ coincides with the KL decomposition based on the stationary response.

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